

Complex Numbers and Quadratic Equations

- A number of the form $a + ib$, where a and b are real numbers and $i = \sqrt{-1}$, is defined as a complex number.
- For the complex numbers $z = a + ib$, a is called the real part (denoted by $\text{Re } z$) and b is called the imaginary part (denoted by $\text{Im } z$) of the complex number z .

Example: For the complex number $z = \frac{-5}{9} + i\frac{\sqrt{3}}{17}$, $\text{Re } z = \frac{-5}{9}$ and $\text{Im } z = \frac{\sqrt{3}}{17}$

- Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are equal if $a = c$ and $b = d$.
- **Addition of complex numbers**

Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ can be added as,
 $z_1 + z_2 = (a + ib) + (c + id) = (a + c) + i(b + d)$

- **Properties of addition of complex numbers:**
 - **Closure Law:** Sum of two complex numbers is a complex number. In fact, for two complex numbers z_1 and z_2 , such that $z_1 = a + ib$ and $z_2 = c + id$, we obtain $z_1 + z_2 = (a + c) + i(b + d)$.
 - **Commutative Law:** For two complex numbers z_1 and z_2 ,

$$z_1 + z_2 = z_2 + z_1.$$

- **Associative Law:** For any three complex numbers z_1 , z_2 and z_3 ,

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3).$$

- **Existence of Additive Identity:** There exists a complex number $0 + i0$ (denoted by 0), called the additive identity or zero complex number, such that for every complex number z , $z + 0 = z$.
- **Existence of Additive Inverse:** For every complex number $z = a + ib$, there exists a complex number $-a + i(-b)$ [denoted by $-z$], called the additive inverse or negative of z , such that $z + (-z) = 0$.

- **Subtraction of complex numbers**

Given any two complex numbers z_1 , and z_2 , the difference $z_1 \ominus z_2$ is defined as
 $z_1 - z_2 = z_1 + (-z_2)$.

- **Multiplication of complex numbers:**

Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ can be multiplied as,
 $z_1 z_2 = (ac - bd) + i(ad + bc)$

- Properties of multiplication of complex numbers:
- **Closure law:** The product of two complex numbers is a complex number.

In fact, for two complex numbers z_1 and z_2 , such that $z_1 = a + ib$ and $z_2 = c + id$, we obtain $z_1 z_2 = (ac - bd) + i(ad + bc)$.

- **Commutative Law:** For any two complex numbers z_1 and z_2 , $z_1 z_2 = z_2 z_1$.
- **Associative Law:** For any three complex numbers z_1 , z_2 and z_3 ,

$$(z_1 z_2) z_3 = z_1 (z_2 z_3).$$

- **Existence of Multiplicative Identity:** There exist a complex number $1 + i0$ (denoted as 1), called the multiplicative identity, such that for every complex numbers z , $z \cdot 1 = z$.
- **Existence of Multiplicative Inverse:** For every non-zero complex number $z = a + ib$ ($a \neq 0, b \neq 0$), we have the complex number

$$\frac{a}{a^2 + b^2} + i \frac{-b}{a^2 + b^2} \quad \left(\text{denoted by } \frac{1}{z} \text{ or } z^{-1} \right), \text{ called the multiplicative inverse of } z,$$

such that $z \frac{1}{z} = 1$.

Example: The multiplicative inverse of the complex number $z = 2 - 3i$ can be found as,

$$z^{-1} = \frac{2}{(2)^2 + (-3)^2} + i \frac{(-3)}{(2)^2 + (-3)^2} = \frac{2}{13} - \frac{3}{13}i$$

- **Distributive Law:** For any three complex numbers z_1 , z_2 and z_3 ,
- $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$
- $(z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3$

- **Division of Complex Numbers**

Given any two complex number z_1 and z_2 , where $z_2 \neq 0$, the quotient $\frac{z_1}{z_2}$ is defined as $\frac{z_1}{z_2} = z_1 \frac{1}{z_2}$

Example: For $z_1 = 1 + i$ and $z_2 = 2 - 3i$, the quotient $\frac{z_1}{z_2}$ can be found as,

$$\begin{aligned}
 \frac{z_1}{z_2} &= \frac{1+i}{2-3i} \\
 &= (1+i) \left(\frac{1}{2-3i} \right) \\
 &= (1+i) \left(\frac{2}{(2)^2+(-3)^2} + i \frac{(-3)}{(2)^2+(-3)^2} \right) \\
 &= (1+i) \left(\frac{2}{13} - \frac{3}{13}i \right) \\
 &= \left[1 \times \frac{2}{13} - 1 \times \left(-\frac{3}{13} \right) \right] + i \left[1 \times \left(-\frac{3}{13} \right) + 1 \times \frac{2}{13} \right] \\
 &= \frac{5}{13} - \frac{1}{13}i
 \end{aligned}$$

- **Property of Complex Numbers**

- For any integer k , $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, $i^{4k+3} = -i$.
- If a and b are negative real numbers, then

- **Modulus of a Complex Number**

The modulus of a complex number $z = a + ib$, is denoted by $|z|$, and is defined as the non-negative real number $\sqrt{a^2 + b^2}$, i.e., $|z| = \sqrt{a^2 + b^2}$.

Example 1: If $z = 2 - 3i$, then $|z| = \sqrt{(2)^2 + (-3)^2} = \sqrt{13}$

- **Conjugate of Complex Number**

The conjugate of a complex number $z = a + ib$, is denoted by $\bar{z}$, and is defined as the complex number $a - ib$, i.e., $\bar{z} = a - ib$.

Example 2: Find the conjugate of $\frac{2}{3+5i}$.

Solution: We have

$$\begin{aligned}
& \frac{2}{3+5i} \\
&= \frac{2}{3+5i} \times \frac{3-5i}{3-5i} \\
&= \frac{2(3-5i)}{(3)^2 - (5i)^2} \\
&= \frac{2(3-5i)}{9-25i^2} \\
&= \frac{6-10i}{9+25} \quad (\because i^2 = -1) \\
&= \frac{6-10i}{34} \\
&= \frac{6}{34} - \frac{10i}{34} \\
&= \frac{3}{17} - \frac{5i}{17}
\end{aligned}$$

Thus, the conjugate of $\frac{2}{3+5i}$ is $\frac{3}{17} + \frac{5i}{17}$.

- Properties of modulus and conjugate of complex numbers:

For any three complex numbers z, z_1, z_2 ,

- $z^{-1} = \frac{\bar{z}}{|z|^2}$ or $z \cdot \bar{z} = |z|^2$
- $|z_1 z_2| = |z_1| |z_2|$
- $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$, provided $|z_2| \neq 0$
- $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$
- $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$

- **Solutions of the quadratic equation when $D < 0$.**

The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a, b, c \in \mathbb{R}$, $a \neq 0$ are given

by $x = \frac{-b \pm \sqrt{D}}{2a}$, where $D = b^2 - 4ac < 0$.

Example: Solve $2x^2 + 3ix + 2 = 0$

Solution: Here $a = 2$, $b = 3i$ and $c = 2$

$$D = b^2 - 4ac$$

$$= (3i)^2 - 4 \times 2 \times 2$$

$$= -9 - 16$$

$$= -25 < 0$$

$$\therefore x = \frac{-b \pm \sqrt{D}}{2a}$$

$$\Rightarrow x = \frac{-3i \pm \sqrt{-25}}{2 \times 2}$$

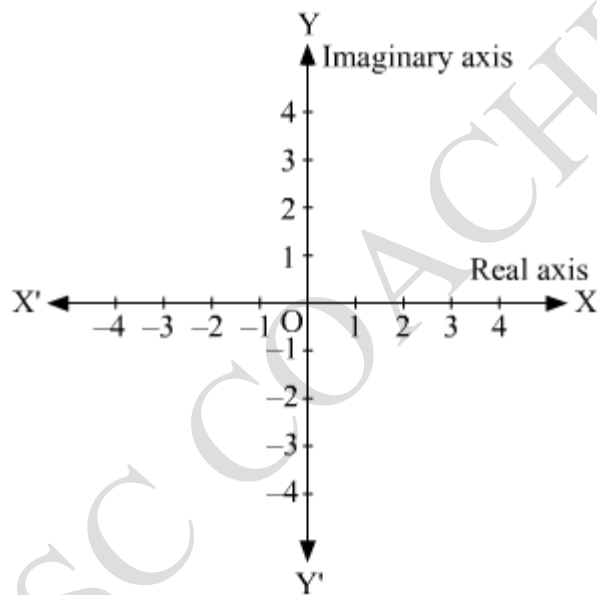
$$\Rightarrow x = \frac{-3i \pm 5i}{4}$$

$$\Rightarrow x = \frac{-3i + 5i}{4} \text{ or } x = \frac{-3i - 5i}{4}$$

$$\Rightarrow x = \frac{2i}{4} \text{ or } x = -\frac{8i}{4}$$

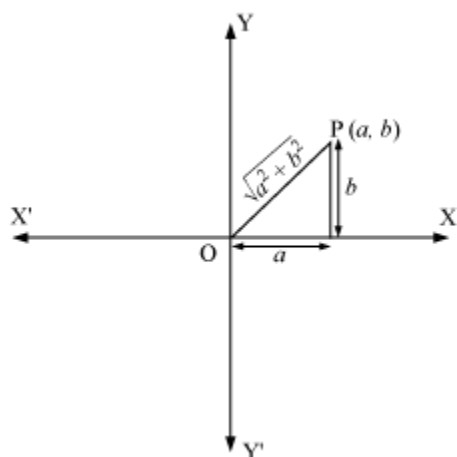
$$\Rightarrow x = \frac{i}{2} \text{ or } x = -2i$$

- **Argand plan:** Each complex number represents a unique point on Argand plane. An Argand plane is shown in the following figure.



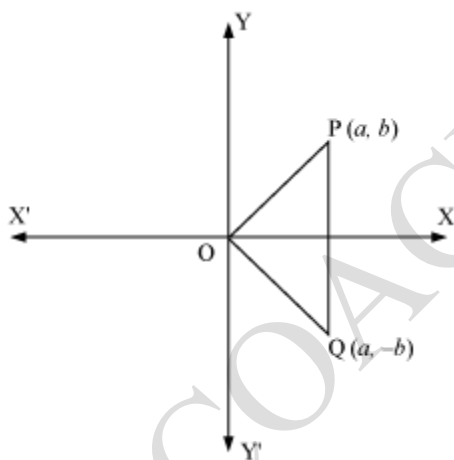
Here, x-axis is known as the **real axis** and y-axis is known as the **imaginary axis**.

- **Complex Number on Argand plane:** The complex number $z = a + ib$ can be represented on an Argand plane as



In this figure, $OP = \sqrt{a^2 + b^2} = |z|$

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- Thus, the modulus of a complex number $z = a + ib$ is the distance between the point $P(x, y)$ and the origin O .
- **Conjugate of Complex Number on Argand plane:**
- The conjugate of a complex number $z = a + ib$ is $\bar{z} = a - ib$, z and $\bar{z}$ can be represented by the points $P(a, b)$ and $Q(a, -b)$ on the Argand plane as



Thus, on the Argand plane, the conjugate of a complex number is the mirror image of the complex number with respect to the real axis.

• Polar representation of Complex Numbers

The polar form of the complex number $z = x + iy$,
is $r(\cos \theta + i \sin \theta)$ where $r = \sqrt{x^2 + y^2}$ (modulus of z)

and $\cos \theta = \frac{x}{r}$, $\sin \theta = \frac{y}{r}$ (θ is known as the argument of z).

The value of θ is such that $-\pi < \theta \leq \pi$, which is called the principle argument of z .

Example 2: Represent the complex number $z = \sqrt{2} - i\sqrt{2}$ in polar form.

Solution: $z = \sqrt{2} - i\sqrt{2}$

Let $\sqrt{2} = r \cos \theta$ and $-\sqrt{2} = r \sin \theta$

By squaring and adding them, we have

$$2 + 2 = r^2 (\cos^2 \theta + \sin^2 \theta)$$

$$\Rightarrow r^2 = 4$$

$$\Rightarrow r = \sqrt{4} = 2$$

Thus,

$$\cos \theta = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$\sin \theta = \frac{-\sqrt{2}}{2} = \frac{-1}{\sqrt{2}} = \sin \left(2\pi - \frac{\pi}{4} \right)$$

$$\Rightarrow \theta = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$$

Thus, the required polar form is $2 \left(\cos \frac{7\pi}{4} + \sin \frac{7\pi}{4} \right)$.